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Nonlinear acoustics in non-uniform infinite and finite layers

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The propagation of weakly nonlinear acoustic waves in a non-uniform medium is treated. It is assumed that the waves are one-dimensional. Non-uniformities arising from variable cross-section and stratification are included. The effect of non-uniformities on unidirectional waves on an infinite interval and resonant waves on a finite interval is discussed for a near-uniform reference state (geometrical acoustics limit) and for stronger non-uniformities in the finite-interval case. Nonlinearities are taken into account up to quadratic and, wherever necessary, cubic order in the wave amplitude.

Unidirectional waves in the geometrical acoustics limit can formally be reduced to the behaviour in a uniform system described by a kinematic wave equation with constant coefficients. For illustration acceleration waves in a weakly non-uniform medium are treated. The resonance case in the geometrical acoustics limit is closely related to resonance in a uniform system so that the methods developed for that situation require only slight modification. For larger influence of non-uniformity the geometrical acoustics limit does not apply and the resonance problem may lead to a Duffing oscillator type of behaviour.

1. Introduction

This paper presents results on nonlinear acoustic wave propagation in a non-uniform medium. The study is motivated by the fact that acoustic wave propagation is inevitably affected by non-uniformity in practical applications or experimental situations so that it is necessary to estimate its influence. There are several references that deal with acoustic wave propagation in non-uniform media. Naturally, the majority of references treats linear waves (Brekhovskikh 1960; Blokhintsev 1946), but there are some that take into account nonlinearity as well (Seymour & Mortell 1975; Lighthill 1978; Pierce 1981; Hayes & Runyan 1970; Varley & Cumberbatch 1970). Keller & Lu Ting (1966) described free nonlinear one-dimensional waves on a finite interval for various mechanical systems (nonlinear strings, beams and gas columns with non-uniformities. In a non-fluid-mechanical context Nayfeh & Mook (1979) use the multiple scale method to derive results in a non-uniform solid nonlinear Hookean slab of variable cross-section and stiffness. Crighton (1992) gives a treatment of unidirectional wave motion in an ideal gas with slowly varying cross-section $A(x)$.

Here the medium is also taken to be an ideal gas for simplicity; this is not an essential restriction and can easily be discarded. The sources of the non-uniformity are a variable cross-section along the direction of propagation and a density stratification imposed, say, by a temperature gradient along the same direction. A case of particular interest is the non-uniformity being weak enough that geometrical acoustics assumptions apply. In this case the gradual change of the non-uniformities is assumed to be small enough for wave reflections due to variable cross-section and/or composition to be neglected; then it makes sense to carry over the concept of unidirectional wave motion in uniform systems and develop a nonlinear extension. This has been described by Lighthill (1978) in a heuristic manner. Here a more formal approach using the multiple scale method (Nayfeh & Mook 1979) is applied in §4 to treat unidirectional wave motion within the approximations of geometrical acoustics. In this context a treatment for acceleration waves is given for illustration.

When the non-uniform medium is of finite extent wave evolution is no longer unidirectional. Most pronounced nonlinear effects are to be expected when the finite system is excited close to resonance since then linear theory predicts large amplitudes and, in fact, fails at exact resonance.

Chester (1964) gives an account of the resonance problem in a uniform system using a theory quadratic in the response amplitude. The crucial feature is the existence of shock discontinuities travelling back and forth in the layer being periodically reflected at both ends. Shock discontinuities exist in a wavelength interval of finite width around the resonance length. The size of this interval depends on the excitation amplitude and the detuning, i.e. the difference between the actual layer width and the resonance width. The same phenomenon appears in a weakly non-uniform system. Keller (1977) investigated an unstratified system (shock tube) with a variable cross-section distribution $A(x) \sim x^{-2}$. For this particular cross-sectional distribution the problem can be reduced to a 'modified Chester' case.

Here the resonance problem is extended to include both stratification and variable cross-section. This study is not restricted to Keller's $A \sim x^{-2}$ case. It turns out that for the resonance problem the role played by the non-uniformities can be caught by just one additional parameter, the admittance of the system (Lighthill 1978). Things become more involved once the geometrical acoustics assumption is dispensed with. The concept of unidirectional wave motion is no longer applicable; however, the resonance problem still makes sense. The result for the resonance problem in this case of 'sufficiently strong' non-uniformity influence is somewhat surprising. Such a system may behave like a (frequency) dispersive one in that there is no such thing as the appearance of shock discontinuities under near-resonance conditions. The system, then, exhibits nonlinear detuning instead, the extent of which primarily depending on the particular distributions of the non-uniformities, among other factors.

In §2 the basic equations are given and put into different forms convenient for treatment of the cases addressed above. Section 3 describes the perturbation procedure for the resonance cases of §5, while §4 treats unidirectional wave motion under the conditions of geometrical acoustics. Evolution equations for density and pressure perturbation and particle velocity are derived for right-running waves in the quadratic approximation. Section 5 investigates the resonances problem in a finite layer, both within the geometrical acoustics approximation and for 'stronger' non-uniformities. The concluding §6 summarizes the results.

2. Basic equations

In this section the basic equations of nonlinear one-dimensional acoustics are presented and manipulated into a form suitable for further treatment. The equation of continuity is (Lighthill 1978)

$$A(x)\dot{\rho}+(\rho u A)_x = 0. \tag{2.1}$$

$A(x)$ denotes the variable cross-section distribution of the stream tube and x is the

Eulerian space coordinate. Partial time differentiation at fixed x is indicated by an overdot and partial differentiation with respect to x at fixed t is expressed by a subscript x; ρ, u are the mass density and particle velocity, respectively. The momentum equation is

$$\rho(\dot{u}+uu_x)+p_x = 0, \tag{2.2}$$

with p the pressure. It is assumed here that, at least in shock-free regions, fluid particles retain their entropy s, so that

$$\dot{s}+us_x = 0. \tag{2.3}$$

Introducing the isentropic velocity of sound, a, by

$$a^2 := (p_\rho)_s, \tag{2.4}$$

equation (2.3) can be replaced by

$$\dot{p}+up_x-a^2(\dot{\rho}+u\rho_x) = 0. \tag{2.5}$$

For convenience ideal gas behaviour is supposed in what follows, so that

$$a^2 = \gamma p/\rho, \tag{2.6}$$

with γ the (constant) isentropic exponent. The restriction to ideal gas behaviour is not essential and can be generalized easily.

Equations (2.1)–(2.3) and (2.5) contain only one spatial coordinate; since they are used to describe the behaviour in a variable two- or three-dimensional-geometry environment, this requires justification. Intuitively it is clear that the variation of the wave motion must be 'sufficiently gradual' in the transverse direction for the one-dimensional formulation to make sense. Specifically the meaning of u, ρ, p and s as averaged quantities is of relevance. Ockendon *et al.* (1993) touch upon this matter in the context of the resonance problem. In their paper they start from the nonlinear wave equation for the velocity potential in a plane wave guide with appropriate boundary conditions at the channel walls. A non-dimensional formulation leads them quite naturally to the conditions under which use of a one-dimensional model that is equivalent to the basic set used here is justified.

Dissipation and body forces have been omitted from the formulation. Now, there is a state of reference in which the system is at rest, i.e.

$$u \equiv 0, \quad \rho = \rho_0(x), \quad p = p_0 = \text{const.}, \quad a = a_0(x). \tag{2.7}$$

Without body forces this state is isobaric. Assume that there is an inherent timescale ω^{-1} dictated by boundary conditions, say, which is used to make time t non-dimensional; also the acoustic lengthscale $a_0(0)\,\omega^{-1}$ is chosen to make x non-dimensional, so that the non-dimensional time $\tilde{t}$ and space coordinate $\tilde{x}$ read

$$\tilde{t} = \omega t, \quad \tilde{x} = \omega x/a_0(0). \tag{2.8}$$

The non-dimensional dependent variables are defined here by

$$\tilde{u}(\tilde{x},\tilde{t}) = u(x,t)/a_0(0), \tag{2.9}$$

$$\tilde{\rho}(\tilde{x},\tilde{t}) = [\rho(x,t)/\rho_0(x)]-1, \tag{2.10}$$

$$\tilde{p}(\tilde{x},\tilde{t}) = [p(x,t)/p_0]-1. \tag{2.11}$$

Using non-dimensional variables from now on it is convenient to omit the tilde symbols in what follows, so that the non-dimensional variants (2.1), (2.2), (2.5) with (2.6) take the form

$$\dot{u}+uu_x+\frac{1}{\gamma}c_0^2\frac{p_x}{1+\rho}=0, \tag{2.12}$$

$$\dot{\rho}+u_x+u(1+\rho)\frac{\mathrm{d}}{\mathrm{d}x}\ln(A\rho_0)+(u\rho)_x=0, \tag{2.13}$$

$$(1+\rho)(\dot{p}+up_x)-\gamma(1+p)\left[\dot{\rho}+u(1+\rho)\frac{\mathrm{d}}{\mathrm{d}x}\ln\rho_0+u\rho_x\right]=0. \tag{2.14}$$

ρ, p, u denote non-dimensional perturbation quantities. The momentum balance contains the square of the variable velocity of sound, $c_0^2(x)$, which is given by

$$c_0^2(x)=\frac{a_0^2(x)}{a_0^2(0)}=\frac{1}{\rho_0(x)}. \tag{2.15}$$

$A(x)$ and $\rho_0(x)$ are non-dimensional from equation (2.12) and have the meaning $A(x)/A(0)$, $\rho_0(x)/\rho_0(0)$ in dimensional terms, respectively. Ordinary differentiation notation $\mathrm{d}/\mathrm{d}x$ is used for the non-uniformity terms.

So far no restrictions on the size of the perturbation quantities ρ, p, u have been imposed. In the next section a perturbation analysis is performed for small but finite perturbation amplitudes. Depending on the order of magnitude of the non-uniformity influence, quadratic and cubic expansions in the perturbation amplitude are appropriate.

3. Perturbation analysis

Two cases of different non-uniformity influence deserve particular attention and are treated below in preparation for the resonance problems of §5.

First, consider the non-uniformity terms in equations (2.12) to (2.14) to be $O(1)$ quantities, i.e. assume $\mathrm{d}\rho_0/\mathrm{d}x, \mathrm{d}A/\mathrm{d}_x \sim O(1)$. Then, with ϵ scaling the perturbation amplitude, which is assumed to be sufficiently small, the following cubic expansion is considered:

$$u(x,t)=\epsilon u_1(x,t;\tau)+\epsilon^2u_2(x,t;\tau)+\epsilon^3u_3(x,t;\tau)+O(\epsilon^4); \tag{3.1}$$

ρ and p are expanded similarly. In the expansion equation (3.1) u_1, u_2, u_3 are considered $O(1)$ functions, namely

$$u_1, u_2, u_3 \sim O(1). \tag{3.2}$$

The same holds for the expansion functions of the ρ and p series. All expansion functions also contain a 'slow-timescale' variable τ defined by

$$\tau := \epsilon^2 t. \tag{3.3}$$

Use of a temporal slow scale variable rather than a spatial slow scale is the appropriate choice for finite-interval-problems.

For infinite-interval-problems this is different, however, for spatially varying non-uniformities where it is essential to apply spatial stretching (Nayfeh & Mook 1979). Substitution of the multiple scale expansion into (2.12)–(2.14) and expanding the time derivative according to

$$\left.\frac{\partial}{\partial t}\right|_x=\left.\frac{\partial}{\partial t}\right|_{x,\tau}+\epsilon^2\left.\frac{\partial}{\partial \tau}\right|_{x,t} \tag{3.4}$$

one arrives at the following perturbation sequence:

$$\dot{\rho}_1 + u_{1x} + u_1 \frac{d}{dx} \ln (A\rho_0) = 0, \tag{3.5}$$

$$\dot{u}_1 + \frac{1}{\gamma} c_0^2 p_{1x} = 0, \tag{3.6}$$

$$\dot{p}_1 - \gamma \left(\dot{\rho}_1 + u_1 \frac{d}{dx} \ln \rho_0 \right) = 0 \tag{3.7}$$

for the $O(\epsilon)$ problem. The overdot has the meaning specified by the first term on the right of equation (3.4).

At $O(\epsilon^2)$ one finds

$$\dot{\rho}_2 + u_{2x} + u_2 \frac{d}{dx} \ln (A\rho_0) = -u_1 \rho_1 \frac{d}{dx} \ln (A\rho_0) - (\rho_1 u_1)_x, \tag{3.8}$$

$$\dot{u}_2 + \frac{1}{\gamma} c_0^2 p_{2x} = -u_1 u_{1x} + \frac{1}{\gamma} c_0^2 \rho_1 p_{1x}, \tag{3.9}$$

$$\dot{p}_2 - \gamma \left(\dot{\rho}_2 + u_2 \frac{d}{dx} \ln \rho_0 \right) = -u_1 p_{1x} + p_1 \dot{p}_1 - \rho_1 \dot{p}_1 + \gamma u_1 \rho_{1x} + \gamma u_1 \rho_1 \frac{d}{dx} \ln (A\rho_0). \tag{3.10}$$

The $O(\epsilon^3)$ problem reads

$$\dot{\rho}_3 + u_{3x} + u_3 \frac{d}{dx} \ln (A\rho_0) = -\rho_{1\tau} - (u_1 \rho_2 + u_2 \rho_1) \frac{d}{dx} \ln (A\rho_0), \tag{3.11}$$

$$\dot{u}_3 + \frac{1}{\gamma} c_0^2 p_{3x} = -u_{1\tau} - (u_1 u_2)_x + \frac{1}{\gamma} c_0^2 (\rho_1 p_{2x} + \rho_2 p_{1x}) - \frac{1}{\gamma} c_0^2 \rho_1^2 p_{1x}, \tag{3.12}$$

$$\begin{aligned} \dot{p}_3 - \gamma \left(\dot{\rho}_3 + u_3 \frac{d}{dx} \ln \rho_0 \right) = & -p_{1\tau} + \gamma \rho_{1\tau} + \gamma (u_1 \rho_{2x} + u_2 \rho_{1x}) - u_1 p_{2x} - u_2 p_{1x} + p_1 \dot{p}_2 \\ & + p_2 \dot{p}_1 - \rho_1 \dot{p}_2 - \rho_2 \dot{p}_1 + \rho_1 p_1 \dot{p}_1 + u_1 \rho_1 p_{1x} + u_1 p_1 p_{1x} \\ & - p_1^2 \dot{p}_1 + \gamma (u_1 \rho_2 + u_2 \rho_1) \frac{d}{dx} \ln \rho_0. \end{aligned} \tag{3.13}$$

This completes the formulation of the first case.

The second case assumes the non-uniformities to be $O(\epsilon)$. To be specific, it is characterized by the assumptions

$$\frac{d^m}{dx^m} \rho_0 = O(\epsilon^m), \tag{3.14}$$

$$\frac{d^m}{dx^m} A = O(\epsilon^m), \quad m = 1, 2, 3, \ldots . \tag{3.15}$$

From assumption (3.14) $d^m c_0 / dx^m$ is also an $O(\epsilon^m)$ quantity. Equations (3.14) and (3.15) are the assumptions of geometrical acoustics (see e.g. Pierce 1981): the lengthscales introduced by the non-uniformities of the medium are assumed to be large compared to the acoustic lengthscale $a_0(0)/\omega$. In other words, on the acoustic lengthscale there is near-uniform behaviour. With this in mind it is convenient to put (2.12)–(2.14) into the following form:

$$\dot{\rho}+u_x = -u\frac{\mathrm{d}}{\mathrm{d}x}\ln(A\rho_0)-(\rho u)_x, \tag{3.16}$$

$$\dot{u}+\frac{1}{\gamma}p_x = -uu_x+\frac{1}{\gamma}\rho p_x-\frac{1}{\gamma}(c_0^2-1)p_x, \tag{3.17}$$

$$\dot{p}-\gamma\dot{\rho} = -up_x-(\rho-p)\dot{p}+\gamma\left[u\frac{\mathrm{d}}{\mathrm{d}x}\ln\rho_0+u\rho_x\right]. \tag{3.18}$$

Equations (3.16)–(3.18) are a quadratic approximation of the 'exact' basic system equations (2.12)–(2.14) in the geometrical acoustics limit with the terms of quadratic order written as the right-hand side of the system. It will turn out that a quadratic expansion is sufficient for the treatment of the resonance problem in §5, for which the previous equations are the basis.

Applying an expansion of the type (3.1) for the dependent variables results in the following sequence:

$$\dot{\rho}_1+u_{1x} = 0, \tag{3.19}$$

$$\dot{u}_1+(1/\gamma)p_{1x} = 0, \tag{3.20}$$

$$\dot{p}_1-\gamma\dot{\rho}_1 = 0. \tag{3.21}$$

The slow timescale τ is now defined to be

$$\tau := \epsilon t, \tag{3.22}$$

and the time-derivative dot in the previous equations denotes time differentiation with both x and τ held fixed as before. One easily recognizes that the first-order equations imply the usual one-dimensional wave equation for u_1,

$$\ddot{u}_1-u_{1xx} = 0, \tag{3.23}$$

with unit phase velocity; equivalent equations hold for ρ_1, p_1.

At the next order one finds

$$\dot{\rho}_2+u_{2x} = -u_1\frac{\mathrm{d}}{\mathrm{d}x}\ln(A\rho_0)-(u_1\rho_1)_x-\rho_{1\tau}, \tag{3.24}$$

$$\dot{u}_2+\frac{1}{\gamma}p_{2x} = -u_1u_{1x}+\frac{1}{\gamma}\rho_1p_{1x}+\frac{1}{\gamma}(1-c_0^2)p_{1x}-u_{1\tau}, \tag{3.25}$$

$$\dot{p}_2-\gamma\dot{\rho}_2 = -u_1p_{1x}+(p_1-\rho_1)\dot{p}_1+\gamma\left[u_1\frac{\mathrm{d}}{\mathrm{d}x}\ln\rho_0+u_1\rho_{1x}\right]. \tag{3.26}$$

In (3.26) no τ-derivative terms occur since they cancel out identically due to the implications of equation (3.21). After some algebra which uses (3.19)–(3.21) repeatedly one arrives at the following inhomogeneous wave equation for u_2:

$$\ddot{u}_2-u_{2xx} = -2\dot{u}_{1\tau}-(1-c_0^2)\ddot{u}_1+\left(u_1\frac{\mathrm{d}}{\mathrm{d}x}\ln A\right)_x-\gamma(\rho_1\rho_{1x})-2(u_1u_{1x}). \tag{3.27}$$

It is worthwhile to give a brief summary of what has been achieved so far and to motivate the next step. Multiple (time) scale expansions in terms of the wave amplitude ϵ were used to expand the basic equations of one-dimensional nonlinear acoustics in a density-stratified, variable-cross-section tube filled by an ideal gas. If the non-uniformities are $O(1)$ quantities a cubic approximation in the amplitude leads to (3.5)–(3.13), whereas under the assumptions made in the geometrical acoustics limit a quadratic expansion results in (3.19)–(3.26), which are to be solved. In both cases initial and/or boundary conditions have to be specified for completion. The forms given above are particularly well suited to the resonance problems; a treatment of the resonance problem on a finite interval is presented later. In the next section unidirectional wave motion on an infinite x-interval is considered.

4. Unidirectional wave motion in the geometrical acoustics limit

The form that we shall start from is slightly different from the one given in (3.16)–(3.18), namely

$$\dot{\rho}+u_x = -u\frac{\mathrm{d}}{\mathrm{d}x}\ln(A\rho_0)-(\rho u)_x, \tag{4.1}$$

$$\dot{u}+\frac{c_0^2}{\gamma}p_x = -uu_x+\frac{c_0^2}{\gamma}\rho p_x, \tag{4.2}$$

$$\dot{p}-\gamma\dot{\rho} = -up_x-(\rho-p)\dot{p}+\gamma\left[u\frac{\mathrm{d}}{\mathrm{d}x}\ln\rho_0+u\rho_x\right]; \tag{4.3}$$

however, the two are equivalent. Substituting the linear approximations

$$p\sim\gamma\rho,\quad \dot{\rho}\sim -u_x,\quad \dot{u}\sim c_0^2\rho_x \tag{4.4a–c}$$

in the right-hand sides of (4.1) and (4.2) leads to

$$\dot{\rho}+u_x = -u\frac{\mathrm{d}}{\mathrm{d}x}\ln(A\rho_0)-(\rho u)_x, \tag{4.5}$$

$$\dot{u}=\frac{c_0^2}{\gamma}p_x = -uu_x+c_0^2\rho\rho_x, \tag{4.6}$$

$$\dot{p}-\gamma\dot{\rho} = \gamma\left[(\gamma-1)\rho\dot{\rho}+u\frac{\mathrm{d}}{\mathrm{d}x}\ln\rho_0\right]. \tag{4.7}$$

The continuity equation (4.5) remains unaffected but applying (4.4b, c) to manipulate the $(\rho u)_x$ term one finally finds

$$\dot{\rho}+u_x = -u\frac{\mathrm{d}}{\mathrm{d}x}\ln(A\rho_0)+\frac{1}{2c_0^2}\dot{u}^2+\tfrac{1}{2}\dot{\rho}^2, \tag{4.8}$$

$$\dot{u}+\frac{c_0^2}{\gamma}p_x = -uu_x+c_0^2\rho\rho_x, \tag{4.9}$$

$$\dot{p}-\gamma\dot{\rho} = \tfrac{1}{2}\gamma(\gamma-1)\dot{\rho}^2+\gamma u\frac{\mathrm{d}}{\mathrm{d}x}\ln\rho_0. \tag{4.10}$$

Introducing the particle displacement $v(x,t)$ by

$$u=\dot{v}, \tag{4.11}$$

it is found that, to quadratic order,

$$\rho = -v_x-v\frac{\mathrm{d}}{\mathrm{d}x}\ln(A\rho_0)+\frac{1}{2c_0^2}\dot{v}^2+\tfrac{1}{2}v_x^2, \tag{4.12}$$

and
$$\frac{p}{\gamma} = -v_x - v\frac{\mathrm{d}}{\mathrm{d}x}\ln A + \frac{1}{c_0^2}\dot{v}^2 + \tfrac{1}{2}\gamma v_x^2. \tag{4.13}$$

In the quadratic terms ρ was replaced by $-v_x$ to derive the last two equations. Substituting them into (4.9) yields the nonlinear wave equation sought:

$$\ddot{v} - c_0^2 v_x = c_0^2\left(v\frac{\mathrm{d}}{\mathrm{d}x}\ln A\right)_x + \frac{1}{c_0}\frac{\mathrm{d}c_0}{\mathrm{d}x}\dot{v}^2 - 2\dot{v}\dot{v}_x - (\gamma-1)\,c_0^2 v_x v_{xx}. \tag{4.14}$$

Since, according to (3.14) and (3.15), $A(x)$, $\rho_0(x)$ and $c_0(x)$ are assumed to be slowly varying, the nonlinear wave equation for $v(x,t)$ may be further simplified to the form

$$\ddot{v} - c_0^2 v_{xx} = c_0^2 v_{xx} = c_0^2 v_x \frac{\mathrm{d}}{\mathrm{d}x}\ln A - 2\dot{v}\dot{v}_x - (\gamma-1)\,c_0^2 v_x v_{xx}, \tag{4.15}$$

in which the terms $v(\mathrm{d}^2/\mathrm{d}x^2)\ln A$ and $(\mathrm{d}c_0/\mathrm{d}_x)\,\dot{v}^2$ have been ignored as they are of cubic order by assumption.

Equation (4.15) describes one-dimensional displacement wave propagation in an ideal gas in a quadratic approximation in the wave amplitude and, simultaneously, in the geometrical acoustic limit, i.e. the non-uniformities are 'sufficiently small'. It allows for wave propagation in both directions. For unidirectional waves running to the right, let

$$v = f\left(t - \int_{(x)} \frac{\mathrm{d}\bar{x}}{c_0(\bar{x})};\, \xi\right). \tag{4.16}$$

Here it is assumed that $f \sim O(\epsilon)$, with ϵ denoting a typical displacement amplitude arising from boundary or initial data. $\xi := \epsilon x$ is a 'slow' space variable. As mentioned in the previous section the use of a spatially stretched coordinate is implied by the spatial variation of the medium (Lighthill 1978). In this context it is convenient to make the 'slowly varying' feature of the non-uniformities explicit by considering A and c_0 as functions of ξ rather than x, i.e. $A = A(\xi))$ and $c_0 = c_0(\xi))$. The unidirectional wave ansatz equation (4.16), after substitution into the wave equation (4.15), then implies

$$-\frac{1}{c_0}\frac{\mathrm{d}c_0}{\mathrm{d}\xi}\dot{f} + 2\dot{f}_\xi + \dot{f}\frac{\mathrm{d}}{\mathrm{d}\xi}\ln A = \frac{\gamma}{c_0^2}\dot{f}\ddot{f}, \tag{4.17}$$

and, finally, identifying $\dot{f}$ with u, yields

$$u_\xi + \tfrac{1}{2}u\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\frac{A}{c_0} - \frac{\gamma+1}{2c_0^2}u\dot{u} = 0. \tag{4.18}$$

For $c_0 = \text{const.}$ equation (4.18) specializes to one given by Crighton (1992). The associated evolution equations for pressure and density perturbations are

$$p_\xi + \tfrac{1}{2}p\frac{\mathrm{d}}{\mathrm{d}\xi}\ln Y - \frac{\gamma+1}{2\gamma c_0}p\dot{p} = 0, \tag{4.19}$$

$$\rho_\xi + \tfrac{1}{2}\rho\frac{\mathrm{d}}{\mathrm{d}\xi}\ln Y - \frac{\gamma+1}{2c_0}\rho\dot{\rho} = 0. \tag{4.20}$$

The quantity Y appearing in (4.19) and (4.20) is referred to as the acoustic admittance of the system (Lighthill 1978). It is an essential quantity, to be met again in §5. Its definition is given as the ratio of the cross-section A and the free wave impedance $\rho_0 c_0$ (see (5.21)).

Equation (4.18) can be transformed to the classical simple wave equation with constant coefficients. To motivate this transformation note that the linear portion of (4.18) implies that $u(A/c_0)^{\frac{1}{2}}$ is constant along the linear characteristics

$$t = \int_{(x)} \frac{\mathrm{d}\bar{x}}{c_0(\bar{x})} = \text{const.};$$

the same holds for $pY^{\frac{1}{2}}$ and also $\rho Y^{\frac{1}{2}}$. With the nonlinearity present this can no longer be true. The linear result, now, suggests the introduction of the new dependent variable

$$\bar{u} := u(A/c_0)^{\frac{1}{2}}, \tag{4.21}$$

so that after substitution of (4.21) into (4.18) the following evolution equation is obtained:

$$\bar{u}_\xi - \frac{\gamma+1}{2c_0\, Y^{\frac{1}{2}}} \bar{u}\dot{\bar{u}} = 0\ ; \tag{4.22}$$

it takes the desired constant coefficient form

$$\bar{u}_\eta - \bar{u}\dot{\bar{u}} = 0, \tag{4.23}$$

when the new independent spatial coordinate $\eta(\xi)$ is introduced according to

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{\gamma+1}{2c_0\, Y^{\frac{1}{2}}}. \tag{4.24}$$

Thus the problem of integrating equation (4.18) for u is analogous to finding the solution of (4.23) for $\bar{u}$. The corresponding solution procedure for $\bar{u}$ is standard (Whitham 1974). $\bar{u}$ is constant along straight line characteristics in the (η, t)-plane, whose inclination depends on the magnitude of the perturbation carried. Thus the rather complicated problem of nonlinear wave propagation in a non-uniform medium can be reduced to an analogous problem in a uniform medium when the assumptions of geometrical acoustics hold. This result has been derived previously by Lighthill (1978) in a more heuristic way. Equations (4.19) and (4.20) for perturbation pressure and density, respectively, are treated in the same manner.

An illustration of the interplay between nonlinearity and non-uniformity is provided by the behaviour of acceleration waves. Introduce

$$t' = t - T(x), \tag{4.25}$$

with

$$T(x) := \int_{x_0}^{x} \frac{1}{c_0(\bar{x})}\, \mathrm{d}\bar{x}, \tag{4.26}$$

where $t' = 0$ denotes the wave front position and x_0 is the wave front initial position. Let $b(\xi)$ be the acceleration along the wave front $t' = 0$, i.e.

$$\bar{u} = b(\xi)\, t'. \tag{4.27}$$

Equation (4.27) implies that u is continuous across the wave front $t' = 0$, but $\dot{u}$ undergoes a finite jump of magnitude $b(\xi)\,(c_0(\xi)/A(\xi))^{\frac{1}{2}}$ along the wave front. Adjacent to the acceleration wave, ahead of the wave front $t' = 0$ there is a state of quiescence. Inserting the ansatz into (4.22) yields

$$\frac{1}{b^2}\frac{\mathrm{d}b}{\mathrm{d}\xi} = \frac{\gamma+1}{2c_0(\xi)\,(Y(\xi))^{\frac{1}{2}}}. \tag{4.28}$$

From (4.28) one concludes that b decreases (increases) for increasing (decreasing) $c_0 Y^{\frac{1}{2}} \equiv (Ac_0^3)^{\frac{1}{6}}$, i.e. for diverging cross-section and positive temperature gradient in the propagation direction the effect of nonlinearity (in this approximation) is weakened by non-uniformity; the nonlinearity effect is enhanced by non-uniformity if propagation is into a narrowing tube that becomes cooler towards the narrowing end. In the intermediate case of Ac_0^3 equalling a constant (cooling towards the diverging end or heating towards the narrowing end), which may be taken as unity, the rate of nonlinear steepening is uninfluenced by non-uniformity.

The picture developed above also makes it plausible that, with non-uniformity present, periodic nonlinear free waves may exist, which is in sharp contrast to the uniform case where they do not (Keller & Lu Ting 1966). If the wave experiences steepening on the way to one reflecting end it will experience flattening on the way back to the other reflecting end, so that the net nonlinearity effect is zero.

5. The resonance problem

Linear theory based on either (3.5)–(3.7) in the case of an $O(1)$ non-uniformity or (3.19)–(3.21) in the limit of geometrical acoustics does not yield a bounded response under resonant conditions. Even with some sort of damping present response amplitudes may become large enough to invalidate the linear theory.

Chester (1964), gave a quadratically nonlinear treatment of the resonance problem in a uniform rigid tube with one end closed and a piston oscillating sinusoidally at the other end that catches the essentials of the phenomena observed in experiments (Lettau 1939; Saenger & Hudson 1960). In a finite frequency band around the resonance frequency, the width of which depends on the excitation amplitude and the detuning (i.e. the deviation from exact resonance), shock waves appear which travel back and forth in the tube, being periodically reflected between the closed end and the oscillating piston. A similar treatment for resonance in a tube with variable cross-section $A(x) \sim x^{-2}$ but without stratification was given by Keller (1977), where the variable-cross-section case could be reduced to a 'modified' Chester case leading to results qualitatively similar to the uniform case.

Here a treatment is given that in the geometrical acoustics limit includes density stratification and variable-cross-section effects not restricted to any particular shapes of $A(x)$ or $\rho_0(x)$ as long as incommensurability of the eigenvalue spectrum of the linearized (non-excited) system is ensured. It will be found that even under these more general conditions Chester's method is applicable. If the admittance distribution is symmetric with respect to the centreline of the system there is even quasi-uniform behaviour at exact resonance.

To be specific, the following boundary conditions are considered:

$$x = 0: \quad u = 0, \tag{5.1}$$

$$x = \lambda - \alpha \sin t: \quad u = \alpha \sin t, \tag{5.2}$$

$$\lambda := \omega L / a_0(0). \tag{5.3}$$

Here, α denotes the excitation amplitude (i.e. the piston Mach number amplitude based on $a_0(0)$) whose relation to the response amplitude ϵ is anticipated here to be

$$\epsilon = \alpha^{\frac{1}{2}}, \tag{5.4}$$

and λ is the non-dimensional system length. The solution sought is required to fulfil the periodicity condition

$$u(x, t) = u(x, t + 2\pi), \tag{5.5}$$

and the mean value condition

$$\int_0^{2\pi} u(x,t)\,\mathrm{d}t = 0. \tag{5.6}$$

With the time variable non-dimensionalized by the excitation circular frequency ω the period of the solution is 2π for all x of interest, see (5.5).

To express the closeness of the system's length λ to the primary resonance length π it is appropriate to write

$$\lambda = \pi + \epsilon\Delta, \quad \Delta = O(1). \tag{5.7}$$

Let Δ be called the 'detuning' from exact resonance $\Delta = 0$. Expanding the boundary-condition equation (5.2) using the representation according to (5.7) and (3.1) for small ϵ with the slow timescale τ obeying (3.22) one finds

$$u_1(\pi, t; \tau) = 0, \tag{5.8}$$

$$u_2(\pi, t; \tau) = \sin t - \Delta u_{1x}(\pi, t; \tau). \tag{5.9}$$

At the closed end, $x = 0$, u_1 vanishes. Thus u_1 can be expressed by

$$u_1(x, t; \tau) = f(t-x; \tau) - f(t+x; \tau) \tag{5.10}$$

as the superposition of an as yet undetermined pair of right- and left-running d'Alembert waves of shape f satisfying $u_1(0, t; \tau) = 0$. The boundary condition (5.8) implies 2π-periodicity in the 'fast' argument $t \pm x$, i.e.

$$f(t \pm x; \tau) = f(t \pm x + 2\pi; \tau). \tag{5.11}$$

At the next order one has to solve the inhomogeneous wave equation (3.27) with ρ_1 given by

$$\rho_1(x, t; \tau) = f(t-x; \tau) + f(t+x; \tau), \tag{5.12}$$

subject to the boundary condition (5.9) and $u_2(0, t; \tau) = 0$. Substituting (5.10) and (5.12) into the right-hand side of (3.27) yields an inhomogeneous wave equation for u_2 with the shape f (to be determined) appearing in the source terms. A particular solution is given below. To derive it note, that, if

$$\ddot{u} - u_{xx} = \frac{\mathrm{d}g(x)}{\mathrm{d}x} f(t \pm x), \tag{5.13}$$

a particular solution is given by

$$u = -\int_0^x g(\xi)\,(t \mp x \pm 2\xi)\,\mathrm{d}\xi. \tag{5.14}$$

This can be verified by substitution. The particular solution fulfils the condition of vanishing u_2 at $x = 0$. After some effort one, eventually, finds the particular solution (using (5.13) and (5.14) wherever necessary) as

$$\begin{aligned} u_2 = {} & -x[f_\tau(t-x;\tau) + f_\tau(t+x;\tau)] + \tfrac{1}{4}(\gamma+1)\,x[f^2(t-x;\tau) + f^2(t+x;\tau)] \\ & -\tfrac{1}{4}(\gamma-3)\,[F(t-x;\tau)\dot{f}(t+x;\tau) - F(t+x;\tau)\dot{f}(t-x;\tau)] \\ & -\int_0^x [\xi - R_0(\xi)]\,[\ddot{f}(t+x-2\xi;\tau) - \ddot{f}(t-x+2\xi;\tau)]\,\mathrm{d}\xi \\ & +\int_0^x \ln A(\xi)\,[\dot{f}(t+x-2\xi;\tau) + \dot{f}(t-x+2\xi;\tau)]\,\mathrm{d}\xi \\ & -\int_0^x \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\ln A(\xi)\right)[f(t+x-2\xi;\tau) - f(t-x+2\xi;\tau)]\,\mathrm{d}\xi, \end{aligned} \tag{5.15}$$

with
$$\rho_0(x) =: \frac{\mathrm{d}R_0(x)}{\mathrm{d}x}, \quad \dot{F} := f. \tag{5.16}$$

The equation determining $f(\cdot\,;\tau)$ follows by substituting (5.10) and (5.15) into the boundary condition at the oscillating piston, (5.9),

$$\begin{aligned}\sin t = &-2\pi f_\tau(t-\pi,\tau) + \tfrac{1}{2}\pi(\gamma+1)\,(f^2(t-\pi,\tau))\\ &-2\left[\Delta - \int_0^\pi (c_0(\xi)-1)\,\mathrm{d}\xi\right]\dot{f}(t-\pi,\tau)\\ &-\frac{1}{2}\int_0^\pi \ln Y(\xi)\,[\dot{f}(t-\pi-2\xi,\tau)+\dot{f}(t-\pi+2\xi,\tau)]\,\mathrm{d}\xi.\end{aligned}$$

The last equation governs the evolution of the nonlinear signal *from the state of rest.* Cox & Mortell (1983, 1985) use this approach to describe resonant gas oscillations and resonant water wave oscillations in shallow water in a uniform environment. The long time response of the system is described by the steady-state version of the previous equation ($\partial/\partial\tau = 0$); for steady state it can be integrated one more time. The resulting equation is then brought into the final form;

$$\cos^2\tfrac{1}{2}\theta + c = (F-\delta)^2 - \frac{1}{2[\pi(\gamma+1)]^{\frac{1}{2}}}\int_0^\pi \ln Y(\xi)\,[F(\theta-2\xi)+F(\theta+2\xi)]\,\mathrm{d}\xi. \tag{5.17}$$

Here
$$\theta := t-\pi \tag{5.18}$$

is the shifted time, and c is a constant of integration, whose determination is part of the solution procedure for (5.17). $F(t)$ is defined as

$$F(t) := \tfrac{1}{2}[\pi(\gamma+1)]^{\frac{1}{2}} f(t), \tag{5.19}$$

and is simply the appropriately redefined f; it should not be confused with the F of (5.16) where it had the meaning of the primitive function of f; it was used only temporarily then. A modified detuning δ appears, which is related to the detuning Δ by

$$\delta := \frac{1}{[\pi(\gamma+1)]^{\frac{1}{2}}}\left[\Delta - \int_0^\pi (c_0(\xi)-1)\,\mathrm{d}\xi\right]. \tag{5.20}$$

δ is an $O(1)$ quantity.

The integral term in (5.17) essentially represents the non-uniformity effect reflected in the appearance of the system's admittance (Lighthill 1978)

$$Y(x) := \frac{A(x)}{\rho_0(x)\,c_0(x)}. \tag{5.21}$$

The geometrical acoustics limit implies $Y(x)$ to be a slowly varying function on the scale of a typical wavelength, which is $O(\pi)$ for the case under consideration.

For convenience it should be mentioned that the manipulations leading to equation (5.17) include the repeated application of the 2π-periodicity of f and F, respectively, see (5.11), and the use of the mean value condition (5.6); the final form of the integral term in (5.17) is arrived at by applying partial integration several times. Finally the trigonometric identity $\cos\theta \equiv 1-2\sin^2\frac{1}{2}\theta$ is used to rewrite the excitation term.

The solution of (5.17) describes the steady-state signal at the oscillating piston. The non-uniformity effect is represented by the modified detuning δ and, more importantly, by the admittance integral. For uniform conditions Y is a constant that can be chosen to be unity without loss of generality. Also, if $A(x)$ and $\rho_0(x)$ vary in such a way that $Y = 1$ the integral term in (5.17) vanishes identically and one directly arrives at Chester's equation (Chester 1964). The constant-Y case means that $A(x)$ and $[\rho_0(x)]^{\frac{1}{2}}$ are proportional for isobaric reference states. In other words, for the resonance problem in the geometrical acoustics limit the effect of the variation of $A(x)$ can be compensated by imposing a negative temperature gradient where $A(x)$ increases and vice versa; a similar situation was found for the behaviour of acceleration waves at the end of §4. The only influence of the non-uniformity apparent, then, is the rather trivial modification of the detuning parameter δ according to (5.20); it reflects the correction of the actual resonance frequency due to the (weak) effect of stratification upon the linear phase velocity.

For uniform systems (5.17) reduces to Chester's classical solution (Chester, 1964), the outstanding feature of which is the existence of shock waves in a δ-band given by $-2/\pi < \delta < 2/\pi$; outside this range the solutions are continuous. It may be suspected that discontinuous solutions of (5.17) also exist with the non-uniformity term included. In fact, a discontinuous solution of (5.17) can be determined very easily in the case $\delta = 0$; it is readily confirmed by substitution that

$$F = \cos\tfrac{1}{2}\theta \quad \text{for} \quad 0 < \theta < 2\pi. \tag{5.22}$$

The solution is to be imagined continued 2π-periodically outside the $(0, 2\pi)$-interval, with the (shifted) shock arrival time θ_s from the mean value condition (5.6) for F given by

$$\theta_s = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots. \tag{5.23}$$

The condition under which the solution according to (5.22) holds turns out to be

$$\int_0^\pi \ln Y(\xi)\cos\xi \, d\xi = 0, \tag{5.24}$$

i.e. $Y(x)$ must be symmetrical about $x = \frac{1}{2}\pi$,

$$Y(x) = Y(\pi - x) \quad \text{or} \quad \frac{A(x)}{A(\pi - x)} = \frac{c_0(\pi - x)}{c_0(x)}, \tag{5.25}$$

respectively. The magnitude of the jump discontinuity at $\theta = \theta_s$, $\|F\|$, is under these circumstances found to be

$$\|F\| = 2, \tag{5.26}$$

and the jump occurs symmetrically about the line $F = 0$. Furthermore, the constant of integration in (5.17), c, is calculated as

$$c = 0. \tag{5.27}$$

For a graphical illustration of this solution the reader is referred to figure 3 of Chester (1964). Thus, for non-uniformities that fulfil the symmetry condition (5.25) and vanishing detuning $\delta = 0$ the existence of a discontinuous solution is illustrated. A remarkable fact about this solution is that although non-uniformity is present it has no influence upon the shape of the signal, if the admittance of the system is symmetric about the middle section. For $\delta \neq 0$ or unsymmetric admittance things become more involved. A statement that still can be made regardless of the particular form of the admittance is that for discontinuous solutions of (5.17) F always jumps symmetrically to the line $F = \delta$, since the non-uniformity integral represents a continuous function of θ even for discontinuous F. The constant of integration, c, appearing in (5.17) has to be determined such that $\min[(F(\theta)-\delta)^2] = 0 \,\forall\, \theta \in [0, 2\pi]$. The mean value condition then serves to fix the temporal shock position θ_s, whereas for continuous solutions the mean value condition is used to fix c.

Ockendon *et al.* (1993) and Keller (1977) give a similar treatment of the resonance problem without stratification, emphasizing the geometrical effects. In different ways they derive an equation that is equivalent to (5.17). The generalization to include stratification is a straightforward matter and the result is that the type of equation one deals with remains unchanged with stratification included in addition to variable geometry. Thus stratification- and variable-geometry-non-uniformity can be combined by the admittance concept in the geometrical acoustics limit.

Progress in solving (5.17) is hard to achieve beyond this point without specifying $Y(x)$. Ockendon *et al.* (1993) take the route of choosing a particular shape for the cross-section distribution. For the particular cross-section shape characterized by parameter κ measuring the 'size' of the geometrical non-uniformity they discuss the division of the (κ, Δ)-plane into shock-free and discontinuous regions; for details the reader is referred to the above paper.

Next, a brief sketch of how to proceed under conditions of $O(1)$ non-uniformities, i.e. when

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln A, \quad \frac{\mathrm{d}}{\mathrm{d}x}\ln \rho_0 = O(1), \tag{5.28}$$

is given; i.e. non-uniformity terms arise at the same perturbation level as the linear terms. The system of field equations to be solved consists of (3.5)–(3.13). The boundary conditions are as before (see (5.1)–(5.3)). The excitation amplitude α of the oscillating piston occurring in (5.2) is now related to the response amplitude ϵ by

$$\epsilon = \alpha^{\frac{1}{3}}. \tag{5.29}$$

The system length λ is assumed to be in an $O(\epsilon^2)$-neighbourhood of the resonance length $\overset{*}{\lambda}$,

$$\lambda = \overset{*}{\lambda} + \epsilon^2 \Delta, \quad \Delta = O(1). \tag{5.30}$$

$\overset{*}{\lambda}$ denotes one out of infinitely many denumerable eigenvalues of the following self-adjoint Sturm–Liouville eigenvalue problem:

$$(A\phi')' + \left[A(\ln A)'' + \frac{A}{c_0^2}\right]\phi = 0, \tag{5.31}$$

$$\phi(0) = \phi(\overset{*}{\lambda}) = 0. \tag{5.32}$$

Primes indicate ordinary differentiation with respect to x. The eigenfunction $\phi(x)$ to the eigenvalue $\overset{*}{\lambda}$ arises as the multiplicative factor of the separation ansatz

$$u_1 = a(\tau)\,\phi(x)\sin(t+\psi(\tau)). \tag{5.33}$$

The solution of equation (5.33) solves the problem

$$(Au_{1x})_x + u_1 A\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln A - \frac{A}{c_{0)}^2}\ddot{u}_1 = 0, \tag{5.34}$$

$$u_1(0, t; \tau) = u_1(\overset{*}{\lambda}, t; \tau) = 0. \tag{5.35}$$

Equation (5.34) is derived by eliminating p_1, ρ_1 from the system of equations (3.5)–(3.5); $a(\tau), \theta(\tau)$ denote slowly varying amplitude and phase with τ defined by (3.3).

At the next order one has to solve (3.8)–(3.10) subject to the boundary conditions

$$u_2(0,t;\tau) = u_2(\overset{*}{\lambda},t;\tau) = 0. \tag{5.36}$$

For what follows it is assumed that, if $\overset{*}{\lambda}$ is an eigenvalue, $2\overset{*}{\lambda}$, and $3\overset{*}{\lambda}$ are not. Then the one-mode-ansatz equation (5.33) proves adequate. The elimination procedure to derive an inhomogeneous form of (5.34) for u_2 is as before. If $2\overset{*}{\lambda}$ is not an eigenvalue then a particular solution is readily found to be of the form

$$u_2 = a^2(\tau)\,\Phi(x)\sin(2t+2\theta(\tau)). \tag{5.37}$$

$\Phi(x)$ can be expressed in terms of the eigenfunction $\phi(x)$ of the first-order problem, but it is not displayed here for brevity.

At third-order secular terms arise on the right-hand sides of (3.11)–(3.13); the boundary conditions of the third-order problem are

$$u_3(0,t;\tau) = 0, \tag{5.38}$$

$$u_3(\overset{*}{\lambda},t;\tau) = \sin t - \Delta u_{1x}(\overset{*}{\lambda},t;\tau). \tag{5.39}$$

The detuning Δ as defined in (5.30) enters the problem on the right-hand side of the last boundary condition. Without going into further detail, if $3\overset{*}{\lambda}$ is not an eigenvalue, the solvability condition for the $O(\epsilon^3)$ problem turns out to be

$$\mathrm{i}\frac{\mathrm{d}B}{\mathrm{d}\tau} - \Delta B + N|B|^2 B = K. \tag{5.40}$$

The complex amplitude $B(\tau)$ is simply related to the real quantities $a(\tau)$, $\theta(\tau)$ by

$$B = a\exp(\mathrm{i}\theta). \tag{5.41}$$

In (5.41) i is the imaginary unit and N, K denote real constants which depends on integrals of products of the eigenfunction ϕ and its first derivative taken over the interval $(0,\overset{*}{\lambda})$; they are not given here for brevity. K is a measure of the external excitation and can be scaled to unity since (5.40) is a two-parameter evolution equation only, i.e. its solution depends on Δ/K and N/K rather than on N, K, Δ individually. N is a parameter which may take either sign. $K = 0$ marks the case of nonlinear free oscillations of the system with the well-known 'Stokes' dependence of the free oscillation frequency and amplitude. $K = 0$ appears as the backbone parabola in the (a,Δ)-plane with its apex at the origin. Steady-state solutions show hard and soft spring behaviour of the Duffing oscillator (Nayfeh & Mook 1979): for $N > 0$ ($N < 0$) there is a bending over to the right (left) of the multivalued response curve in the (a,Δ)-plane with stable and unstable branches.

Recapitulating, in the geometrical acoustics limit it was found that the geometrical effects and non-uniformities of stratification could be combined by making use of the concept of admittance: one result was that the system behaves quasi-uniformly for unidirectional waves for constant admittance. Quasi-uniform behaviour is also found if the admittance is distributed evenly with respect to the midpoint of the system for exact resonance. For $O(1)$ non-uniformities the concept of admittance looses its strength. Nonlinearity and geometrical and stratification effects interplay and contribute to N rather individually. One may speculate that, contrary to the previous situation, it may then happen that the parameter N disappears for particular distributions of cross-section and stratification. A change of sign of N may occur even for a uniform system as, for instance, in the resonant water-wave sloshing problem investigated by Ockendon, Ockendon & Johnson (1986) with frequency and amplitude dispersion interaction determining the sign of N according to the choice of system parameters (undisturbed depth etc.). If that occurs a cubic expansion in the response amplitude ϵ is not sufficient. Although the $N = 0$ case has not yet been investigated, it is obvious how to proceed: the expansion has to be driven to fifth order in ϵ in (3.1) with

$$\epsilon = (\alpha)^{\frac{1}{5}}, \tag{5.42}$$

replacing (5.29) to eventually arrive at an evolution equation of the form

$$\mathrm{i}\frac{\mathrm{d}B}{\mathrm{d}\tilde{\tau}} - \tilde{\Delta}B + \tilde{N}|B|^4 B = \tilde{K}, \tag{5.43}$$

$$\epsilon^4 \tilde{\Delta} = \lambda - \overset{*}{\lambda}, \quad \tilde{\Delta} = O(1), \tag{5.44}$$

$$\tilde{\tau} = \epsilon^4 t, \tag{5.45}$$

the constants $\tilde{N}$, $\tilde{K}$ being real.

The essential prerequisite for (5.40) or (5.43) to be valid is the incommensurability of the eigenvalue spectrum of the Sturm–Liouville system equations (5.31), (5.32). Ockendon *et al.* (1993) distinguish several cases of partial commensurability and discuss implications qualitatively. A case that belongs to the class of problems represented by (5.40) and illustrates the point is Chester's (1991) work on resonance in a spherical cavity; the spectrum of the linearized spherical problem is incommensurate and the steady-version of (5.40) describes the trans-resonant regime. No attempt has been made to find conditions on $A(x)$, $\rho_0(x)$ ensuring eigenvalue spectrum incommensurability. The quantity that presents the most pronounced inconvenience with respect to analytical treatment is the mass density distribution in the reference state $\rho_0(x)$. A reasonable $c_0^2(x)$ distribution would be a straight line reflecting a constant temperature gradient along the tube axis and a spherical, cylindrical or exponential $A(x)$-distribution. No representability of the eigenfunctions of the corresponding system (5.31), (5.32) in terms of (tabulated) functions could be found. So one has to resort to numerical integration for further treatment, i.e. the ultimate determination of the constants K, N. That is beyond the scope of the present paper which is rather, aimed at contrasting the different features of the resonance response with $O(\epsilon)$ and $O(1)$ non-uniformities.

Keller's (1977) treatment of the variable-cross-section resonance problem takes an interesting 'intermediate' position between the $O(\epsilon)$ and $O(1)$ non-uniformity-resonance cases: although his non-uniformity influence is $O(1)$, the pertinent equation he is led to is not (5.40) but an equation equivalent to (5.17); also the response-excitation relation (5.4) applies in his case rather than (5.29). The reason for this behaviour is that for $A(x) \sim x^{-2}$ the eigenvalue spectrum of the linearized system is fully commensurate; in fact, Keller deliberately chose the cross-section distribution in such a way as to ensure commensurability of the eigenvalues. Then all modes resonate simultaneously for the primary mode being excited as in the case of a uniform system.

6. Concluding remarks

Two problems of weakly nonlinear acoustic wave propagation in non-uniform media have been treated above.

First, unidirectional pulses are considered in the limit of geometrical acoustics with nonlinearity taken into account up to quadratic order in the wave amplitude. It is assumed that within this approximation both the nonlinearity and the non-uniformity effects are of the same order of magnitude. The shape of the unidirectional wave experiences distortion due to nonlinearity and non-uniformity. Mathematically the problem can be reduced to a kinematic or simple wave equation with constant coefficients that is easily solved by the method of characteristics. The solution implies that the signal evolves along characteristics according to linear theory with the shape of the characteristics being determined by nonlinearity. The effects of the mutual influence of nonlinearity and non-uniformity are demonstrated by looking at the behaviour of acceleration waves with the result that non-uniformity may be enhancing or weakening amplitude dispersion.

Second, externally excited waves on the finite interval are considered with nonlinear effects primarily pronounced close to resonance. If nonlinearity and non-uniformity effects are assumed to be of the same order of magnitude as before discontinuous solutions may occur for near-resonant conditions. Weak periodic shock waves dissipate the energy that is pumped into the system by external harmonic excitation. Imagining the discontinuous signal to be decomposed into its Fourier components suggests that all eigenmodes of the linearized system contribute to and participate in generating the discontinuous response. For sufficiently weak non-uniformities shock formation cannot be prevented if the system is close enough to resonance.

The result is markedly different for a system with $O(1)$ non-uniformity in near-resonance. Including nonlinearity up to cubic order in the wave amplitude it turns out that the bounding mechanism for the response amplitude is then nonlinear detuning to which the only excited mode contributes by interacting nonlinearly with itself.

Unlike a uniform system, the eigenfrequencies of a non-uniform finite system are in general not integer multiples of the primary frequency. Assuming incommensurability of the eigenvalue spectrum Keller & Lu Ting (1966) have shown that nonlinear free waves are periodic on the finite interval; with non-uniformity present periodic solutions exist with their frequency depending on the magnitude of the wave amplitude. The energy stays primarily confined to the resonating mode interacting with itself. With external periodic excitation present such a system reacts by nonlinear detuning, in contrast to the nearly uniform system (geometrical acoustics) where a discontinuity develops that ultimately dissipates the perturbation. The confinement of the energy to one mode also accounts for the fact that the response is much larger, i.e. $O(\epsilon^{\frac{1}{3}})$ for $O(1)$ non-uniformities rather than $O(\epsilon^{\frac{1}{2}})$ for $O(\epsilon)$ non-uniformities with all modes resonating and participating in the response.

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